数理解析研究所講究録
第 1782 巻 2012 年 122-134

Continuous limit of random walks and its application to approximation of nonlinear PDEs

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1 Introduction

This paper is a summary of the preprints [8] and [9], where scaling limit of random walks is investigated and it is applied to approximation theories of nonlinear PDEs of hyperbolic types.

Let $\gamma=\{\gamma^{k}\}_{k=0,1,2,\cdots},\gamma^{0}=0$ be the one-dimensional random walk on the rescaled space $\Delta x\mathbb{Z}:=\{x_{m}:=m\Delta x\,|\,m\in \mathbb{Z}\},\Delta x>0$ defined by the symmetric transition probability $\rho(\gamma^{k}=x_{m}\,;\,\gamma^{k+1}=x_{m}\pm\Delta x)=1/2$ and $w_{\Delta}=\{w_{\Delta}(t)\}_{t\geq 0}$ be the stochastic process given by the linear interpolation of γ between each $[t_{k}, t_{k}+\Delta t]$, where $t_{k}:=k\Delta t\in\Delta t\mathbb{Z}_{\geq 0},\Delta t>0$. It is well known as *the law of large numbers* that, for the limit $\Delta:=(\Delta x, \Delta t)\rightarrow 0$ under hyperbolic scaling $\Delta t/\Delta x\equiv 1$, the distribution of w_{Δ} converges weakly to the δ-measure, or equivalently w_{Δ} converges to $w_{0}(t)\equiv 0$ locally uniformly in probability. It is also well known as Donsker's theorem that, for the limit $\Delta=(\Delta x, \Delta t)\rightarrow 0$ under diffusive scaling $\Delta t/\Delta x^{2}\equiv 1$, the distribution of w_{Δ} converges weakly to Wiener measure, or equivalently there exist processes $\hat{w}_{\Delta}$ and Brownian motion B on a probability space $(S, \mathcal{S}, P)$ such that the distributions of $\hat{w}_{\Delta}, w_{\Delta}$ are identical and $\hat{w}_{\Delta}(\omega)$ converge locally uniformly to $B(\omega)$ with probability 1. This fact is based on *the central limit theorem.*

There is large literature on the application of scaling limit of random walks to various fields. Here we study space-time continuous limit of space-time inhomogeneous random walks for $\Delta=(\Delta x, \Delta t)\rightarrow 0$ *under hyperbolic scaling* $0<\lambda_{0}\leq\Delta t/\Delta x=\lambda\leq\lambda_{1}$ *with fixed constants* λ_{0} *and* λ_{1} and apply it to the Lax-Friedrichs finite difference approximation of entropy solutions of scalar conservation laws.

We deal with the random walks $\gamma=\{\gamma^{k}\}_{k=0,1,2,\cdots}$, $\gamma^{0}=0$ defined by the following transition probabilities which are allowed to be far from a homogeneous one:

$$\rho(\gamma^{k}=x_{m}\,;\,\gamma^{k+1}=x_{m}\pm\Delta x):=\frac{1}{2}\pm\frac{1}{2}\lambda\xi(t_{k}, x_{m}),$$

where $\xi:(\Delta t\mathbb{Z}_{\geq 0})\times(\Delta x\mathbb{Z})\rightarrow[-\lambda^{-1}, \lambda^{-1}]$ is a deterministically given function. Note

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that since transition probabilities are inhomogeneous, the law of large number does not always hold and the study of continuous limit is much more complicated.

The Lax-Friedrichs scheme is one of the oldest, simplest and most universal techniques of computing PDEs. There is the huge literature on the scheme as well as many other schemes. We investigate the Lax-Friedrichs scheme applied to inviscid hyperbolic scalar conservation laws *in terms of scaling limit of random walks and calculus of variations.* This approach is quite different from the usual functional analytic argument with a priori estimates.

2 Continuous limit of random walks

We formulate our random walks precisely. Take an arbitrary $T > 0$ and $\Delta = (\Delta x, \Delta t)$. We will vary Δ under the condition $0 < \lambda_0 \le \lambda = \Delta t/\Delta x \le \lambda_1$ with fixed constants λ_0 and λ_1. Let $K \in \mathbb{N}$ be such that $t_K \in (T - \Delta t, T]$. $m(x), k(t)$ denote the integers m, k for which we have $x \in [x_m, x_m + 2\Delta x), t \in [t_k, t_k + \Delta t)$ for $x \in \mathbb{R}, t \ge 0$. We set the following:

$$X^k := \{x_m \mid -k \le m \le k,\ m+k = \text{even}\}\ (k \in \mathbb{Z}_{\ge 0}), \quad G^K := \bigcup_{0 \le k < K} \{t_k\} \times X^k,$$

$$\xi : G^K \ni (t_k, x_m) \mapsto \xi_m^k \in [-\lambda^{-1}, \lambda^{-1}],$$

$$\bar{\bar{\rho}} : G^K \ni (t_k, x_m) \mapsto \bar{\bar{\rho}}_m^k := \frac{1}{2} + \frac{1}{2}\lambda \xi_m^k \in [0,1], \quad \bar{\rho} := 1 - \bar{\bar{\rho}},$$

$$\gamma : \{0, 1, 2 \cdots, K\} \ni k \mapsto \gamma^k \in X^k, \ \gamma^0 = 0, \ \gamma^{k+1} - \gamma^k = \pm \Delta x\ ,$$

$$\Omega^k : \text{ the family of } \gamma|_{\le k} \text{ (the restriction of } \gamma \text{ for } \{0,1,2,\cdots,k\}).$$

We regard $\bar{\bar{\rho}}_m^k$ as a transition probability from (t_k, x_m) to $(t_k + \Delta t, x_m + \Delta x)$ and $\bar{\rho}_m^k$ from (t_k, x_m) to $(t_k + \Delta t, x_m - \Delta x)$. We still use the notation γ for each element of Ω^k. We define the density of each path $\gamma \in \Omega^k$ as

$$\mu^k(\gamma) := \prod_{0 \le k' < k} \rho(\gamma^{k'}, \gamma^{k'+1}),$$

where $\rho(\gamma^{k'}, \gamma^{k'+1}) = \bar{\bar{\rho}}^k_{m(\gamma^{k'})}$ (respectively $\bar{\rho}^k_{m(\gamma^{k'})}$) if $\gamma^{k'+1} - \gamma^{k'} = \Delta x$ $(-\Delta x)$. The density $\mu^k(\gamma)$ yields the probability measure of Ω^k, namely the probability of $A \subset \Omega^k$ is given by $\sum_{\gamma \in A} \mu^k(\gamma)$. In particular we pay our attention to the probability measure of $\Omega_\Delta := \Omega^K$ given by $\mu_\Delta := \mu^K$. We introduce the following:

$$\bar{\xi}^k := \sum_{\gamma \in \Omega_\Delta} \mu_\Delta(\gamma) \xi^k_{m(\gamma^k)}, \quad \rho_+^k := \sum_{\gamma \in \Omega_\Delta} \mu_\Delta(\gamma) \bar{\bar{\rho}}^k_{m(\gamma^k)}, \quad \rho_-^k := \sum_{\gamma \in \Omega_\Delta} \mu_\Delta(\gamma) \bar{\bar{\rho}}^k_{m(\gamma^k)},$$

$$\eta(\gamma) : \{0,1,2,\cdots,K\} \ni k \mapsto \eta^k(\gamma) \in \mathbb{R}, \quad \eta^k(\gamma) := \sum_{0 \le k' < k} \xi^{k'}_{m(\gamma^{k'})} \Delta t, \quad \gamma \in \Omega_\Delta,$$

$$\bar{\gamma} : \{0,1,2,\cdots,K\} \ni k \mapsto \bar{\gamma}^k \in \mathbb{R}, \qquad \bar{\gamma}^k := \sum_{\gamma \in \Omega_\Delta} \mu_\Delta(\gamma) \gamma^k,$$

$$\sigma^k := \sum_{\gamma \in \Omega_\Delta} \mu_\Delta(\gamma) |\gamma^k - \bar{\gamma}^k|^2, \qquad d^k := \sum_{\gamma \in \Omega_\Delta} \mu_\Delta(\gamma) |\gamma^k - \bar{\gamma}^k|,$$

$$\tilde{\sigma}^k := \sum_{\gamma \in \Omega_\Delta} \mu_\Delta(\gamma) |\gamma^k - \eta^k(\gamma)|^2, \qquad \tilde{d}^k := \sum_{\gamma \in \Omega_\Delta} \mu_\Delta(\gamma) |\gamma^k - \eta^k(\gamma)|.$$

We remark that $d^k \leq \sqrt{\sigma^k}$ and $\tilde{d}^k \leq \sqrt{\tilde{\sigma}^k}$. The following recurrence formulas hold:

Theorem 2.1. *1.* $\bar{\gamma}^{k+1} = \bar{\gamma}^k + \bar{\xi}^k \Delta t, \quad \bar{\gamma}^0 = 0.$

2. $\sigma^{k+1} = \sigma^k + 4\rho_+^k \rho_-^k \Delta x^2 + 4 \sum_{\gamma \in \Omega_\Delta} \mu_\Delta(\gamma) \bar{\rho}^k_{m(\gamma^k)}(\gamma^k - \bar{\gamma}^k)\Delta x, \quad \sigma^0 = 0.$

3. $\tilde{\sigma}^{k+1} = \tilde{\sigma}^k + 4 \sum_{\gamma \in \Omega_\Delta} \mu_\Delta(\gamma) \bar{\rho}^k_{m(\gamma^k)} \bar{\rho}^k_{m(\gamma^k)} \Delta x^2, \quad \tilde{\sigma}^0 = 0.$

4. In particular, we have

$$\tilde{\sigma}^k \leq \frac{t_k}{\lambda}\Delta x, \quad \tilde{d}^k \leq \sqrt{\frac{t_k}{\lambda}\Delta x}. \tag{2.1}$$

We remark that the variance σ^k does not necessarily tend to 0. In fact, consider a dis continuous function $\xi(t_k, x_m) := \varepsilon$ (respectively 0, $-\varepsilon$) for $x_m > 0$ ($x_m = 0$, $x_m < 0$) with $\varepsilon > 0$. Then the random walk has the average 0. Direct calculation yields the estimate $\sigma^k \geq (d^k)^2 \geq \varepsilon^2 t_k^2$. Furthermore $p_0^k \sim (1 - \lambda^2\varepsilon^2)^{k/2}$ for large k, which makes the distribution $\{p^k_{m(x)}\}_{x \in X^k}$ split into two parts.

Theorem 2.2. *Suppose that ξ is Lipschitz around $\bar{\gamma}$ with respect to x, namely there exists $\theta > 0$ such that for $\xi_*^k := \xi^k_{m(\bar{\gamma}^k)} + \frac{\xi^k_{m(\bar{\gamma}^k)+2} - \xi^k_{m(\bar{\gamma}^k)}}{2\Delta x}(\bar{\gamma}^k - x_{m(\bar{\gamma}^k)})$, the estimate $|\xi_m^k - \xi_*^k| \leq \theta|x_m - \bar{\gamma}^k|$ holds for all k. Then we have*

$$\sigma^k \leq \frac{e^{4\theta t_k}}{4\theta\lambda}\Delta x.$$

Therefore if ξ satisfies a $\Delta = (\Delta x, \Delta t)$-independent Lipschitz condition, then the variance goes to zero and we have the law of large numbers.

Theorem 2.3. *Consider a sequence of continuous functions $\xi_\Delta(t,x) : [0,T] \times [-\frac{T}{\lambda_0}, \frac{T}{\lambda_0}] \to [-\lambda_1^{-1}, \lambda_1^{-1}]$ which is Lipschitz with respect to x with a Lipschitz constant θ independent of Δ and converges uniformly to ξ_0 as $\Delta \to 0$. Let w_0 be the solution of the ODE $w_0'(t) = \xi(t, w_0(t))$, $w_0(t) = 0$. Then, taking $\xi_m^k := \xi_\Delta(t_k, x_m)$ for each fixed Δ, we have*

1. $\displaystyle\sum_{\gamma \in \Omega_\Delta} \mu_\Delta(\gamma) \left(\sum_{0 \leq k < K} |\gamma^k - \bar{\gamma}^k|^2 \Delta t \right) \leq T \frac{e^{4\theta T}}{4\theta\lambda}\Delta x.$

2. $\displaystyle\sum_{\gamma \in \Omega_\Delta} \mu_\Delta(\gamma) \left(\max_{0 \leq k \leq K} |\eta^k(\gamma) - \bar{\gamma}^k| \right) \leq 2\theta T \sqrt{\frac{e^{4\theta T}}{4\theta\lambda}\Delta x}.$

3. The linear interpolation of $\bar{\gamma}^k$, denoted by $\bar{\gamma}_\Delta$, converges uniformly to w_0 as $\Delta \to 0$.

Let $\mathcal{W}$ be the set of all continuous functions $f : [0,T] \to \mathbb{R}$ with the C^0-norm. We introduce the stochastic processes w_Δ, $\tilde{w}_\Delta : \Omega_\Delta \to \mathcal{W}$ which are the linear interpolations of γ, $\eta(\gamma)$. We remark that all the sample paths of w_Δ, $\tilde{w}_\Delta$ are Lipschitz with a common Lipschitz constant independent of Δ and ξ. The distributions of w_Δ, $\tilde{w}_\Delta$, as probability measures of $\mathcal{W}$, are denoted by $P_\Delta = P_\Delta(\cdot;\xi)$, $\tilde{P}_\Delta = \tilde{P}_\Delta(\cdot;\xi)$. Theorem 2.1, 2.2 and 2.3

imply the following basic limit theorems on the asymptotics of $P_\Delta = P_\Delta(\cdot;\xi)$ and $\tilde{P}_\Delta = \tilde{P}_\Delta(\cdot;\xi)$ for $\Delta = (\Delta x, \Delta t) \to 0$ under hyperbolic scaling $0 < \lambda_0 \le \lambda = \Delta t/\Delta x \le \lambda_1$: The results which hold for any ξ and therefore for any transition probabilities are the following:

Theorem 2.4. *1. For each uniformly continuous function $\mathcal{L} : \mathcal{W} \to \mathbb{R}$, there exists a number $\varepsilon(\Delta, \mathcal{L}) > 0$ which is independent of ξ and tends to 0 as $\Delta \to 0$ such that*

$$\left| \int_{\mathcal{W}} \mathcal{L}(f) P_\Delta(df) - \int_{\mathcal{W}} \mathcal{L}(f) \tilde{P}_\Delta(df) \right| \le \varepsilon(\Delta, \mathcal{L}).$$

2. For each sequence ξ_j, which is not necessarily convergent, and $\Delta_j \to 0$, the sets of probability measures $\{P_{\Delta_j}(\cdot;\xi_j)\}_j$ and $\{\tilde{P}_{\Delta_j}(\cdot;\xi_j)\}_j$ are relatively compact.

Next we impose a Δ-independent Lipschitz condition on ξ.

Theorem 2.5. *Consider a sequence of continuous functions $\xi_\Delta(t,x) : [0,T]\times[-\frac{T}{\lambda_0}, \frac{T}{\lambda_0}] \to [-\lambda_1^{-1}, \lambda_1^{-1}]$ which is Lipschitz with respect to x with a Lipschitz constant θ independent of Δ and converges uniformly to ξ_0 as $\Delta \to 0$. Let w_0 be the solution of the ODE $w_0'(t) = \xi_0(t, w_0(t))$, $w_0(t) = 0$. Then, for $\xi(t_k, x_m) := \xi_\Delta(t_k, x_m)$ with each fixed Δ, we have*

1. *$w_\Delta \to w_0$, $\tilde{w}_\Delta \to w_0$ uniformly in probability as $\Delta \to 0$.*
2. *$P_\Delta \to \delta_{w_0}$, $\tilde{P}_\Delta \to \delta_{w_0}$ weakly as $\Delta \to 0$, where δ_{w_0} is the probability measure of $\mathcal{W}$ supported by $\{w_0\}$.*

3 Variational approach to entropy solutions and viscosity solutions

Before applying the results of the previous section, we recall the variational approach to entropy solutions and viscosity solutions. We consider initial value problems of the inviscid hyperbolic scalar conservation law

$$(3.1) \qquad \begin{cases} u_t + H(x,t,c+u)_x = 0 & \text{in } \mathbb{T}\times(0,T], \\ u(x,0) = u(x) \in L^\infty(\mathbb{T}) & \text{on } \mathbb{T}, \quad \int_{\mathbb{T}} u^0(x)dx = 0, \end{cases}$$

where c is a parameter varying within an interval $[c_0, c_1]$ and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the standard torus. The assumptions for the flux function H are the following (A1)-(A4):

$$\text{(A1) } H(x,t,p) : \mathbb{T}^2\times\mathbb{R}\to\mathbb{R},\ C^2 \quad \text{(A2) } H_{pp} > 0 \quad \text{(A3) } \lim_{|p|\to+\infty} \frac{H(x,t,p)}{|p|} = +\infty.$$

By (A1)-(A3), we have the Legendre transform $L(x,t,\xi)$ of $H(x,t,\cdot)$, which is now given by

$$L(x,t,\xi) = \sup_{p\in\mathbb{R}} \{\xi p - H(x,t,p)\}$$

and satisfies

(A1)' $L(x,t,\xi) : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$, C^2 (A2)' $L_{\xi\xi} > 0$ (A3)' $\lim_{|\xi|\to+\infty} \frac{L(x,t,\xi)}{|\xi|} = +\infty$.

The last assumption is

(A4) There exists $\alpha > 0$ such that $|L_x| \le \alpha(|L| + 1)$.

Throughout this paper, $\mathbb{T}$-dependency is identified with $\mathbb{R}$-dependency with $\mathbb{Z}$-periodicity and $\mathbb{T}$ with $[0,1)$. (A1) and (A2) are standard in the theories of conservation laws. (A3) is necessary, when we introduce a variational approach stated below to our problems. (A4) is used for derivation of boundedness of minimizers of the variational problems. We remark that the whole space setting is also available with additional assumptions for H required for variational techniques.

The problems (3.1) appear not only in continuum mechanics but also in Hamiltonian and Lagrangian dynamics generated by H and L [4], [6], [3]. In the latter case the periodic setting is standard. It is sometimes very convenient to introduce initial value problems of Hamilton-Jacobi equations which are equivalent to (3.1)

$$\text{(3.2)} \qquad \begin{cases} v_t + H(x,t,c+v_x) = h(c) & \text{in } \mathbb{T} \times (0,T], \\ v(x,0) = v^0(x) \in Lip(\mathbb{T}) & \text{on } \mathbb{T}, \end{cases}$$

where $h(c) : [c_0, c_1] \to \mathbb{R}$ is a continuous function. As usual, we consider (3.1) and (3.2) in the class of generalized solutions called entropy solutions and viscosity solutions respectively. Such solutions exist in $C^0((0,T]; L^\infty(\mathbb{T}))$ and $Lip(\mathbb{T} \times (0,T])$. If $u^0 = v^0_x$, then the entropy solution u of (3.1) and the viscosity solution v of (3.2) satisfy $u = v_x$. From now on we always assume that $u^0 = v^0_x$. One of the central achievements in the analysis of (3.1) and (3.2) is that they are closely related to the deterministic calculus of variations: The value of v at each point (x,t) is given by

$$\text{(3.3)} \qquad v(x,t) = \inf_{\gamma\in AC,\ \gamma(t)=x} \left\{ \int_0^t L^c(\gamma(s), s, \gamma'(s))ds + v_0(\gamma(0)) \right\} + h(c)t,$$

where AC is the family of absolutely continuous curves $\gamma : [0,t] \to \mathbb{R}$ and $L^c(x,t,\xi) := L(x,t,\xi) - c\xi$ is the Legendre transform of $H(x,t,c+\cdot)$ (see e.g. [1]). We can find a minimizing curve γ^* of (3.3), which is a C^2-solution of the Euler-Lagrange equation associated with the Lagrangian $L^c(x,t,\xi)$. If the point (x,t) is a regular point of v (i.e. there exists $v_x(x,t)$), then the value $u(x,t)$ is given by

$$\text{(3.4)} \qquad u(x,t) = \int_0^t L^c_x(\gamma^*(s), s, \gamma^{*\prime}(s))ds + u_0(\gamma^*(0)).$$

We remark that, since v is Lipschitz, almost every points are regular. The representation formula (3.3) is the strong tool not only in the analysis of (3.1) and (3.2) but also in many applications of them to other fields such as optimal controls and dynamical systems.

It should be noted that *the variational approach to (3.1) and (3.2) based on (3.3) and (3.4) also contributes approximation theories of (3.1) and (3.2) by the vanishing viscosity method and the finite difference method.* The first case is announced by Fleming [5] and the latter case is the theme of this paper.

First we recall the results of Fleming. Let us consider initial value problems of

$$\text{(3.5)} \qquad u^\nu_t + H(x,t,c+u^\nu)_x = \nu u^\nu_{xx},$$

$$\text{(3.6)} \qquad v^\nu_t + H(x,t,c+v^\nu_x) = h(c) + \nu v^\nu_{xx} \qquad (\nu > 0)$$

with the same setting as (3.1) and (3.2). The solutions u^ν and v^ν are also related to calculus of variations which are not deterministic but stochastic: The value of v^ν at each point (x,t) is given by

$$(3.7)\quad v^\nu(x,t) = \inf_{\xi^\nu \in C^1} E\left[\int_0^t L^c(\gamma^\nu(s), s, \xi^\nu(\gamma^\nu(s), s))ds + v_0(\gamma^\nu(0))\right] + h(c)t,$$

where E stands for the expectation with respect to the Wiener measure and γ^ν is a solution of the stochastic ODE

$$(3.8)\qquad d\gamma^\nu(s) = \xi^\nu(\gamma^\nu(s), s)ds + \sqrt{2\nu}dB(t-s), \quad \gamma^\nu(t) = x.$$

Here B is the standard Brownian motion. There exists the unique minimizing vector field $\xi^{\nu*}$ of (3.7). The value $u^\nu(x,t)$ is given by

$$(3.9)\qquad u^\nu(x,t) = E\left[\int_0^t L^c_x(\gamma^{\nu*}(s), s, \xi^{\nu*}(\gamma^{\nu*}(s), s))ds + u_0(\gamma^{\nu*}(0))\right],$$

where $\gamma^{\nu*}$ is a solution of (3.8) with $\xi^\nu = \xi^{\nu*}$. It is proved from a stochastic and variational point of view that, for $\nu \to 0+$, v^ν converges uniformly to v with the error $O(\sqrt{\nu})$ and u^ν converges pointwise to u except for points of discontinuity of u. In particular, u^ν converges uniformly to u without an arbitrarily small neighborhood of shocks. The proof indicates how the stochastic variational formula (3.7) and (3.9) tend to the deterministic ones (3.3) and (3.4). Asymptotics of γ^ν for $\nu \to 0$ plays a central role, where γ^ν converge to characteristic curves of u and v. Fleming's approach yields much information and concrete pictures of the vanishing viscosity method. In particular we can see how the parabolicity disappears to be hyperbolic.

In [9], the author establishes a stochastic and variational approach to the finite difference method with the Lax-Friedrichs scheme, which holds the advantages of Fleming's approach. We discretize the equation of (3.1) by the Lax-Friedrichs scheme:

$$(3.10)\quad \frac{u^{k+1}_{m+1} - \frac{(u^k_m + u^k_{m+2})}{2}}{\Delta t} + \frac{H(x_{m+2}, t_k, c + u^k_{m+2}) - H(x_m, t_k, c + u^k_m)}{2\Delta x} = 0.$$

We can find a difference equation which approximates the equation of (3.2) and is equivalent to (3.10) in the sense that $u^k_m = (v^k_{m+1} - v^k_{m-1})/2\Delta x$:

$$(3.11)\qquad \frac{v^{k+1}_m - \frac{(v^k_{m-1} + v^k_{m+1})}{2}}{\Delta t} + H(x_m, t_k, c + \frac{v^k_{m+1} - v^k_{m-1}}{2\Delta x}) = h(c).$$

We present stochastic calculus of variations associated with (3.11), which yields representation formulas of v^k_{m+1} and u^k_m similar to (3.7) and (3.9). The stochastic structure of the Lax-Friedrichs scheme is characterized by the space-time inhomogeneous random walks in $\Delta x\mathbb{Z} \times \Delta t\mathbb{Z}$ given in the previous section, instead of (3.8), whose probability measures are no longer related to the Winer measure. This is the main difficulty of our arguments. We need the asymptotics for $\Delta = (\Delta x, \Delta t) \to 0$ of the random walks with arbitrary transition probabilities under *hyperbolic scaling* $0 < \lambda_0 \le \Delta t/\Delta x \le \lambda_1$. It is interesting to note that, under diffusive scaling $\Delta x^2/\Delta t = 2\nu > 0$, the solutions of (3.10) and (3.11) converge to these of (3.5) and (3.6), and the continuous limit of a certain class of random walks is the Brownian motion or some diffusion processes. Our approach also yields much information and concrete pictures of the finite difference method with the Lax-Friedrichs scheme. In particular *we can see how the "parabolicity" due to numerical viscosity disappears to be hyperbolic in terms of the law of large numbers.* Here we point out several new points of our approach:

(1) The stability of the Lax-Friedrichs scheme for arbitrary $T>0$, namely the $\Delta x,\Delta t$-independent boundedness of u^k_m, is verified.

(2) The convergence of u^k_m to u is proved in a framework of the pointwise convergence, where u^k_m tends to the representative element of $u\in L^1$ given by (3.4). In particular the uniform convergence, except neighborhoods of shocks with arbitrarily small measure, is available.

(3) The uniform convergence of v^k_{m+1} to v with an error $O(\sqrt{\Delta x})$ is proved from a stochastic and variational viewpoint.

(4) The approximation of (backward) characteristic curves of (3.1) and (3.2) and its convergence are verified.

The Lax-Friedrichs approximation of entropy solutions (also with other schemes) is basically based on the L^1-framework with a priori estimates, where $\Delta x,\Delta t$-independent boundedness of both u^k_m and its total variation must be verified e.g. [7], [2], [10]. Our stochastic and variational approach is quite different from this with simpler proofs.

4 Stochastic and variational approach to the Lax-Friedrichs scheme

Let N,K be natural numbers. The mesh size $\Delta=(\Delta x,\Delta t)$ is defined by $\Delta x:=(2N)^{-1}$ and $\Delta t:=(2K)^{-1}$. Set $\lambda:=\Delta t/\Delta x$, $x_m:=m\Delta x$ for $m\in\mathbb{Z}$ and $t_k:=k\Delta t$ for $k=0,1,2,\cdots$. For $x\in\mathbb{R}$ and $t>0$, the notation $m(x),k(t)$ denote the integers m,k for which $x\in[x_m,x_m+2\Delta x), t\in[t_k,t_k+\Delta t)$. Let $(\Delta x\mathbb{Z})\times(\Delta t\mathbb{Z}_{\geq 0})$ be the set of all (x_m,t_k) and

$$\mathcal{G}_{even}\subset(\Delta x\mathbb{Z})\times(\Delta t\mathbb{Z}_{\geq 0}),\qquad \mathcal{G}_{odd}\subset(\Delta x\mathbb{Z})\times(\Delta t\mathbb{Z}_{\geq 0})$$

be the set of all (x_m,t_k) with $k=0,1,2,\cdots$ and $m\in\mathbb{Z}$ with $m+k$ =even, odd. We call $\mathcal{G}_{even}$, $\mathcal{G}_{odd}$ the even grid, odd grid. We consider the discretization of (3.1) by the Lax-Freidrichs scheme in $\mathcal{G}_{even}$:

$$(4.1)\quad\begin{cases}\dfrac{u^{k+1}_{m+1}-\frac{(u^k_m+u^k_{m+2})}{2}}{\Delta t}+\dfrac{H(x_{m+2},t_k,c+u^k_{m+2})-H(x_m,t_k,c+u^k_m)}{2\Delta x}=0,\\[2ex] u^0_m=u^0_\Delta(x_m),\quad u^k_{m\pm 2N}=u^k_m,\end{cases}$$

where

$$(4.2)\qquad u^0_\Delta(x):=\frac{1}{2\Delta x}\int_{x_m-\Delta x}^{x_m+\Delta x}u^0(y)dy\quad\text{for } x\in[x_m-\Delta x,x_m+\Delta x).$$

Note that $\sum_{\{m \mid 0 \le m < 2N,\, m+k = even\}} u_m^k \cdot 2\Delta x$ is conservative with respect to k and is zero for u^0 with the average zero. Now we consider a discrete version of (3.2) in $\mathcal{G}_{odd}$:

$$
(4.3) \qquad \begin{cases} \dfrac{v_m^{k+1} - \frac{(v_{m-1}^k + v_{m+1}^k)}{2}}{\Delta t} + H(x_m, t_k, c + \dfrac{v_{m+1}^k - v_{m-1}^k}{2\Delta x}) = h(c), \\ v_{m+1}^0 = v_\Delta^0(x_{m+1}), \quad v_{m+1\pm 2N}^k = v_{m+1}^k, \end{cases}
$$

where v_Δ^0 is a function which converges to v^0 uniformly as $\Delta \to 0$. We introduce the following notation:

$$
D_t w_m^{k+1} := \frac{w_m^{k+1} - \frac{w_{m-1}^k + w_{m+1}^k}{2}}{\Delta t}, \quad D_x w_{m+1}^k := \frac{w_{m+1}^k - w_{m-1}^k}{2\Delta x}.
$$

As an assumption similar to $u^0 = v_x^0$, we also assume that

$$
(4.4) \qquad v_\Delta^0(x) := v^0(0) + \int_0^x u_\Delta^0(y)dy.
$$

Note that $u_\Delta^0 \to u^0$ in L^1 and $v_\Delta^0 \to v^0$ uniformly with $\| v_\Delta^0 - v^0 \|_{C^0} \le \| u^0 \|_{L^\infty} \cdot 2\Delta x$, as $\Delta \to 0$. The two problems (4.1) and (4.3) are equivalent under (4.2) and (4.4):

Proposition 4.1. *Let u_m^k and v_{m+1}^k be the solutions of (4.1) and (4.3) with (4.2) and (4.4). Then we have $D_x v_{m+1}^k = u_m^k$ and we can construct v_{m+1}^k from u_m^k.*

We introduce space-time inhomogeneous backward random walks in $\mathcal{G}_{odd}$ which are required by the Lax-Friedrichs scheme. They are slightly different from the ones introduced in Section 2. However the asymptotic properties are the same. For each point $(x_n, t_{l+1}) \in \mathcal{G}_{odd}$, we consider backward random walks γ which starts from x_n at t_{l+1} and move by $\pm\Delta x$ in each backward time step:

$$
\gamma = \{\gamma^k\}_{k=0,1,\cdots,l+1}, \quad \gamma^{l+1} = x_n, \quad \gamma^{k+1} - \gamma^k = \pm\Delta x.
$$

More precisely, we set the following:

$$
\begin{aligned}
& X^k := \{x \mid x_n - (l+1-k)\Delta x \le x \le x_n + (l+1-k)\Delta x, \ (x, t_k) \in \mathcal{G}_{odd}\}, \\
& G := \bigcup_{1 \le k \le l+1} (X^k \times \{t_k\}) \subset \mathcal{G}_{odd}, \\
& \xi : G \ni (x_m, t_k) \mapsto \xi_m^k \in [-\lambda^{-1}, \lambda^{-1}], \quad \lambda = \Delta t / \Delta x, \\
& \bar{\bar{\rho}} : G \ni (x_m, t_k) \mapsto \bar{\bar{\rho}}_m^k := \frac{1}{2} - \frac{1}{2}\lambda \xi_m^k \in [0,1], \ \ \bar{\rho} := 1 - \bar{\bar{\rho}}, \\
& \gamma : \{0, 1, 2, \cdots, l+1\} \ni k \mapsto \gamma^k \in X^k, \ \gamma^{l+1} = x_n, \ \gamma^{k+1} - \gamma^k = \pm\Delta x \ , \\
& \Omega : \ \text{the family of } \gamma.
\end{aligned}
$$

We regard $\bar{\bar{\rho}}_m^k$ (respectively $\bar{\rho}_m^k$) as a transition probability from (x_m, t_k) to $(x_m + \Delta x, t_k - \Delta t)$ (from (x_m, t_k) to $(x_m - \Delta x, t_k - \Delta t)$). Note that this definition of transition probabilities is different from that in Section 2. We control the transition of the random walks by ξ, which plays a velocity-like role in G. We define the density of each path $\gamma \in \Omega$ as

$$
\mu(\gamma) := \prod_{0 < k \le l+1} \rho(\gamma^k, \gamma^{k-1}),
$$

where $\rho(\gamma^k,\gamma^{k-1}) = \bar{\bar{\rho}}^k_{m(\gamma^k)}$ (respectively $\bar{\rho}^k_{m(\gamma^k)}$) if $\gamma^k - \gamma^{k-1} = -\Delta x$ (Δx). The density $\mu(\cdot) = \mu(\cdot;\xi)$ yields a probability measure of Ω, namely

$$prob(A) = \sum_{\gamma\in A} \mu(\gamma;\xi) \quad \text{for } A \subset \Omega.$$

The expectation with respect to this probability measure is denoted by $E_{\mu(\cdot;\xi)}$, namely for a random variable $f : \Omega \to \mathbb{R}$

$$E_{\mu(\cdot;\xi)}[f(\gamma)] := \sum_{\gamma\in\Omega} \mu(\gamma;\xi) f(\gamma).$$

Set $\Gamma^k_m := \{\gamma \in \Omega \mid \gamma^k = x_m\}$ and $p^k_m := \sum_{\gamma\in\Gamma^k_m} \mu(\gamma)$. We observe the following lemma, which follows from the definition of random walks.

Lemma 4.2. *1.* $\sum_{x\in X^k} p^k_{m(x)} = 1$. *Hence* $\{p^k_{m(x)}\}_{x\in X^k}$ *yields a probability of* X^k.

2. $p^k_m = \sum_{\gamma\in\Gamma^k_m} \mu^k(\gamma)$, *where* $\mu^k(\gamma) := \prod_{k<k'\le l+1} \rho(\gamma^{k'},\gamma^{k'-1})$.

3. $p^k_m = p^{k+1}_{m-1}\bar{\bar{\rho}}^{k+1}_{m-1} + p^{k+1}_{m+1}\bar{\rho}^{k+1}_{m+1}$, *where* $\bar{\bar{\rho}}^{k+1}_{m\pm1}, \bar{\rho}^{k+1}_{m\pm1} = 0$ *if* $x_{m\pm1} \notin X^{k+1}$.

We represent the approximate solutions by the random walks and functionals given by L^c, the Legendre transform of $H(x,t,c+\cdot)$. From now on we assume the following:

Assumption. *Suppose (A1)-(A4). Let* $T > 0$ *be arbitrarily fixed. The parameter* c *varies within* $[c_0, c_1]$. *Initial datas are bounded:* $\| u^0 \|_{L^\infty} = \| v^0_x \|_{L^\infty} \le r$, $\| v \|_{C^0} \le r$.

First of all we see the following proposition, assuming also that there exists a solution u^k_m of (4.1) which satisfies the stability condition called the CFL-condition

$$|H_p(x_m, t_k, c + u^k_m)| < \lambda^{-1} \quad (\lambda = \Delta t/\Delta x).$$

This is informative, because a proof indicates how the Lax-Friedrichs scheme reveals the stochastic and variational structure. The proof also implies that the proposition holds only with the assumptions (A2) and (A3):

Proposition 4.3. *Suppose that we have the solution* v^k_m *of (4.3) for which* $u^k_m := D_x v^k_{m+1}$ *satisfies the CFL-condition for all* m *and* $k = 0,1,2,\cdots,k^*$. *Then* v^k_{m+1} *is represented for each* n *and* $0 < l+1 \le k^*$ *as*

$$v^{l+1}_n = \inf_\xi E_{\mu(\cdot;\xi)}\Big[\sum_{0<k\le l+1} L^c(\gamma^k, t_{k-1}, \xi^k_{m(\gamma^k)})\Delta t + v^0_\Delta(\gamma^0)\Big] + h(c)t_{l+1}. \tag{4.5}$$

The minimizing velocity field ξ^* *is unique and given by*

$$\xi^{*k+1}_m = H_p(x_m, t_k, c + D_x v^k_{m+1}).$$

Proof. Fix $\xi : G \to [-\lambda^{-1}, \lambda^{-1}]$ arbitrarily. It follows form the difference equation (4.3) and the property of the Legendre transform that

$$\begin{aligned}
v_n^{l+1} &= \frac{v_{n-1}^l + v_{n+1}^l}{2} - H(x_n, t_l, c + D_x v_{n+1}^l)\Delta t + h(c)\Delta t \\
&= \{\xi_n^{l+1} \cdot (c + D_x v_{n+1}^l) - H(x_n, t_l, c + D_x v_{n+1}^l)\}\Delta t - c\xi_n^{l+1}\Delta t \\
&\quad + (\frac{1}{2} + \frac{1}{2}\lambda\xi_n^{l+1})v_{n-1}^l + (\frac{1}{2} - \frac{1}{2}\lambda\xi_n^{l+1})v_{n+1}^l + h(c)\Delta t \\
&\le L^c(x_n, t_l, \xi_n^{l+1})\Delta t + (\frac{1}{2} + \frac{1}{2}\lambda\xi_n^{l+1})v_{n-1}^l + (\frac{1}{2} - \frac{1}{2}\lambda\xi_n^{l+1})v_{n+1}^l + h(c)\Delta t,
\end{aligned}$$

where the equality holds, if and only if $\xi_n^{l+1} = H_p(x_n, t_l, c + D_x v_{n+1}^l) \in (-\lambda^{-1}, \lambda^{-1})$. Similarly we have

$$\begin{aligned}
v_{n-1}^l &\le L^c(x_{n-1}, t_{l-1}, \xi_{n-1}^l)\Delta t + (\frac{1}{2} + \frac{1}{2}\lambda\xi_{n-1}^l)v_{n-2}^{l-1} + (\frac{1}{2} - \frac{1}{2}\lambda\xi_{n-1}^l)v_n^{l-1} + h(c)\Delta t, \\
v_{n+1}^l &\le L^c(x_{n+1}, t_{l-1}, \xi_{n+1}^l)\Delta t + (\frac{1}{2} + \frac{1}{2}\lambda\xi_{n+1}^l)v_n^{l-1} + (\frac{1}{2} - \frac{1}{2}\lambda\xi_{n+1}^l)v_{n+2}^{l-1} + h(c)\Delta t,
\end{aligned}$$

where the equality holds, if and only if $\xi_{n\pm1}^l = H_p(x_{n\pm1}, t_{l-1}, c + D_x v_{n\pm1+1}^{l-1}) \in (-\lambda^{-1}, \lambda^{-1})$. Hence we get

$$v_n^{l+1} \le \sum_{l \le k \le l+1} \left(\sum_{x \in X^k} p_{m(x)}^k L^c(x, t_{k-1}, \xi_{m(x)}^k) \right) \Delta t + \sum_{x \in X^{l-1}} p_{m(x)}^{l-1} v_{m(x)}^{l-1} + h(c)(t_{l+1} - t_{l-1}).$$

Continuing this process, we obtain

$$v_n^{l+1} \le \sum_{0 < k \le l+1} \left(\sum_{x \in X^k} p_{m(x)}^k L^c(x, t_{k-1}, \xi_{m(x)}^k) \right) \Delta t + \sum_{x \in X^0} p_{m(x)}^0 v_{m(x)}^0 + h(c) t_{l+1}.$$

The equality holds, if and only if $\xi_m^k = H_p(x_m, t_{k-1}, c + D_x v_{m+1}^{k-1}) \in (-\lambda^{-1}, \lambda^{-1})$. By Lemma 4.2, we see that the first and second term of the right hand side, denoted by A_1 and A_2, are changed into

$$\begin{aligned}
A_1 &= \sum_{0 < k \le l+1} \Big\{ \sum_{x \in X^k} \Big(\sum_{\gamma \in \Omega_{m(x)}^k} \mu(\gamma; \xi) \Big) L^c(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^k) \Big\} \Delta t \\
&= \sum_{0 < k \le l+1} \Big(\sum_{\gamma \in \Omega} \mu(\gamma; \xi) L^c(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^k) \Big) \Delta t \\
&= \sum_{\gamma \in \Omega} \mu(\gamma; \xi) \Big(\sum_{0 < k \le l+1} L^c(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^k) \Delta t \Big), \\
A_2 &= \sum_{x \in X^0} \Big(\sum_{\gamma \in \Omega_{m(x)}^0} \mu(\gamma; \xi) \Big) v_{m(\gamma^0)}^0 = \sum_{\gamma \in \Omega} \mu(\gamma; \xi) v_{m(\gamma^0)}^0.
\end{aligned}$$

ξ is arbitrary and we conclude (4.5). □

Next we remove the assumption of the existence of v_{m+1}^k with the CFL-condition.

Theorem 4.4. *There exists $\lambda_1 > 0$ (depending on T, $[c_0, c_1]$ and r, but independent of Δ) for which we have the following:*

1. *For any* $\Delta = (\Delta x, \Delta t)$ *with* $\lambda = \Delta t/\Delta x < \lambda_1$, *the expectation of functionals for each* n *and* $0 < l+1 < k(T)$

$$(4.6) \qquad E_{\mu(\cdot;\xi)}\Big[\sum_{0<k\le l+1} L^c(\gamma^k, t_{k-1}, \xi^k_{m(\gamma^k)})\Delta t + v^0_\Delta(\gamma^0)\Big] + h(c)t_{l+1}$$

has the infimum denoted by E^{l+1}_n *with respect to* $\xi : G \to [-\lambda^{-1}, \lambda^{-1}]$. *The infimum* E^{l+1}_n *is attained by* ξ^* *which satisfies* $|\xi^*| < \lambda_1^{-1}$.

2. *Define* v^k_{m+1} *for each* m *and* $0 \le k < k(T)$ *as* $v^0_{m+1} := v^0_\Delta(x_{m+1})$, $v^k_{m+1} := E^k_{m+1}$. *Then, for each* n *and* $0 < l+1 < k(T)$, *the minimizing velocity field* ξ^* *which yields* E^{l+1}_n *satisfies*

$$L^c_\xi(x_m, t_k, {\xi^*}^{k+1}_m) = D_x v^k_{m+1} \Leftrightarrow {\xi^*}^{k+1}_m = H_p(x_m, t_k, c + D_x v^k_{m+1}).$$

3. v^k_{m+1} *satisfies (4.3) for* $0 \le k < k(T)$.

Existence and compactness of the minimizer ξ^* is proved by means of (A4) and variational techniques. This theorem immediately leads to one of our main results:

Theorem 4.5. *There exists* $\lambda_1 > 0$ *(depending on* T, $[c_0, c_1]$ *and* r, *but independent of* Δ*) such that for any* $\Delta = (\Delta x, \Delta t)$ *with* $\lambda = \Delta t/\Delta x < \lambda_1$ *we have the solution* u^k_m *of (4.1) which satisfies up to* $k = k(T)$

$$|H_p(x_m, t_k, c + u^k_m)| \le \lambda_1^{-1} < \lambda^{-1} \quad \textit{(CFL-condition)}.$$

Next we "represent" the solution u^k_m of (4.1).

Theorem 4.6. *Let* ξ^* *be the minimizer for* E^{l+1}_n *and* $\mu(\cdot;\xi^*), \gamma, \Omega$ *be for* E^{l+1}_n. *Let* $\tilde{\xi}^*$ *be the minimizer for* E^{l+1}_{n+2} *and* $\tilde{\mu}(\cdot;\tilde{\xi}^*), \tilde{\gamma}, \tilde{\Omega}$ *be for* E^{l+1}_{n+2}. *Then* u^{l+1}_{n+1} *satisfies for each* n *and* $0 < l+1 < k(T)$

$$(4.7) \quad u^{l+1}_{n+1} \le E_{\mu(\cdot;\xi^*)}\Big[\sum_{0<k\le l+1} L^c_x(\gamma^k, t_{k-1}, {\xi^*}^k_{m(\gamma^k)})\Delta t + u^0_\Delta(\gamma^0 + \Delta x)\Big] + O(\Delta x),$$

$$(4.8) \quad u^{l+1}_{n+1} \ge E_{\tilde{\mu}(\cdot;\tilde{\xi}^*)}\Big[\sum_{0<k\le l+1} L^c_x(\tilde{\gamma}^k, t_{k-1}, \tilde{\xi}^{*k}_{m(\tilde{\gamma}^k)})\Delta t + u^0_\Delta(\tilde{\gamma}^0 - \Delta x)\Big] + O(\Delta x),$$

where $O(\Delta x)$ *stands for a number of* $(-\theta\Delta x, \theta\Delta x)$ *with* $\theta > 0$ *independent of* Δx.

We present convergence results of the stochastic and variational approach to the Lax-Friedrichs scheme. We always take the limit $\Delta = (\Delta x, \Delta t) \to 0$ under hyperbolic scaling $0 < \lambda_0 \le \lambda = \Delta t/\Delta x < \lambda_1$. We say that a point $(x,t) \in \mathbb{T} \times (0,T]$ is a regular point, if there exists $v_x(x,t)$. Note that regular points are nothing but points of continuity of $u = v_x$ and almost every points are regular. The minimizing curve of $v(x,t)$ is unique, if (x,t) is regular.

Theorem 4.7. *Let* v_Δ *be the linear interpolation of the approximate solution* v^k_{m+1}. *Then* v_Δ *converges uniformly to the viscosity solution of* v *in* $\mathbb{T} \times [0,T]$. *In particular, we have an error estimate: There exists* $\beta > 0$ *independent of* $\Delta = (\Delta x, \Delta t)$ *such that*

$$\| v_\Delta - v \|_{C^0} \le \beta\sqrt{\Delta x}.$$

This result is consistent with the earlier literature. However the argument is based on the different viewpoint that the random walks become deterministic and our stochastic calculus of variations tend to the deterministic ones as $\Delta = (\Delta x, \Delta t) \to 0$ due to the results of Section 2. The estimate (2.1) plays an essential role.

Theorem 4.8. *Let $(x,t) \in \mathbb{T} \times (0,T]$ be a regular point, (x_n, t_{l+1}) be a point of $[x - 2\Delta x, x + 2\Delta x) \times [t - \Delta t, t + \Delta t)$ and $\gamma^* : [0,t] \to \mathbb{R}$ be the minimizing curve for $v(x,t)$. Let $\gamma_\Delta : [0,t] \to \mathbb{R}$ be the linear interpolation of the random walk γ generated by the minimizing velocity field ξ^* for E_n^{l+1}. Then*

$$\gamma_\Delta \to \gamma^* \text{ uniformly in probability as } \Delta = (\Delta x, \Delta t) \to 0.$$

In particular, the average of γ_Δ converges uniformly to γ^* as $\Delta = (\Delta x, \Delta t) \to 0$.

The minimizing curve γ^* is the genuine backward characteristic curves of v and u starting from (x,t). Therefore the Lax-Friedrichs scheme turns out to approximate not only PDE solutions but also their characteristic curves. If the minimizer ξ^* satisfies the $\Delta = (\Delta x, \Delta t)$-independent Lipschitz condition, Theorem 4.8 is immediately derived from Theorem 2.5. However this is not true, because the entropy solution is discontinuous in general. Nevertheless we can prove the theorem with the aid of variational techniques.

Theorem 4.9. *Let u_Δ be the step function derived from u_m^k, namely $u_\Delta(x,t) = u_m^k$ for $(x,t) \in [x_m - \Delta x, x_m + \Delta x) \times [t_k, t_k + \Delta t)$. Then for each regular point $(x,t) \in \mathbb{T} \times [0,T]$*

$$u_\Delta(x,t) \to u(x,t) \text{ as } \Delta = (\Delta x, \Delta t) \to 0.$$

In particular, u_Δ converges uniformly to u on $(\mathbb{T} \times [0,T]) \setminus \Theta$, where Θ is a neighborhood of the set of points of singularity of u with an arbitrarily small measure.

This convergence result is stronger than the one derived from the usual L^1-framework in the following sense: The approximate solution u_Δ converges pointwise to the particular representative element of $u \in L^1$ which is the derivative of the corresponding viscosity solution and is represented as (3.4). Theorem 4.9 is proved with Theorem 4.6 and Theorem 4.8, namely the right hand side of both (4.7) and (4.8) converge to (3.4).

References

[1] P. Cannarsa and C. Sinestrari, Semiconcave functions, Hamilton-Jacobi equations and optimal control, Birkhäuser (2004).

[2] M. G. Crandall and Majda, Monotone difference approximations for scalar conservation laws, Math. Comp. **34** (1980), No. 149, 1-21.

[3] W. E, Aubry-Mather theory and periodic solutions of the forced Burgers equation, Comm. Pure Appl. Math. **52** (1999), No. 7, 811-828.

[4] A. Fathi, Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens, (French) [A weak KAM theorem and Mather's theory of Lagrangian systems] C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), No. 9, 1043-1046.

[5] W. H. Fleming, The Cauchy problem for a nonlinear first order partial differential equation, J. Differential Equations **5** (1969), 515-530.

[6] H. R. Jauslin, H. O. Kreiss and J. Moser, On the forced Burgers equation with periodic boundary conditions, Proc. Sympos. Pure Math. **65** (1999), 133-153.

[7] O. A. Oleinik, Discontinuous solutions of nonlinear differential equations, A. M. S. Transl. (ser. 2) **26** (1957), 95-172.

[8] K. Soga, Space-time continuous limit of random walk with hyperbolic scaling, preprint.

[9] K. Soga, Stochastic and variational approach to the Lax-Friedrichs scheme, preprint.

[10] E. Tadmor, The large-time behavior of the scalar, genuinely nonlinear Lax-Friedrichs scheme, Math. Comp. **43** (1984), No. 168, 353-368.